Singularity-Invariant Families of Line-Plane 5-S$\underline{P}$U Platforms

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***Abstract*—A 5-S$\underline{P}$U robot with collinear universal joints is well suited to handle an axisymmetric tool, since it has five controllable degrees of freedom, and the remaining one is a free rotation around the tool. The kinematics of such a robot also having coplanar spherical joints has previously been studied as a rigid subassembly of a Stewart–Gough platform, which has been denoted a line-plane component. Here, we investigate how to move the leg attachments in the base and the platform without altering the robot's singularity locus. By introducing the so-called 3-D space of leg attachments, we prove that there are only three general topologies for the singularity locus corresponding to the families of quartically, cubically, and quadratically solvable 5-S$\underline{P}$U robots. The members of the last family have only four assembly modes, which are obtained by solving two quadratic equations. Two practical features of these quadratically solvable robots are the large manipulability within each connected component and the fact that, for a fixed orientation of the tool, the singularity locus reduces to a plane.**

***Index Terms*—Gough–Stewart platforms, kinematics singularities, manipulator design, parallel manipulators, robot kinematics.**

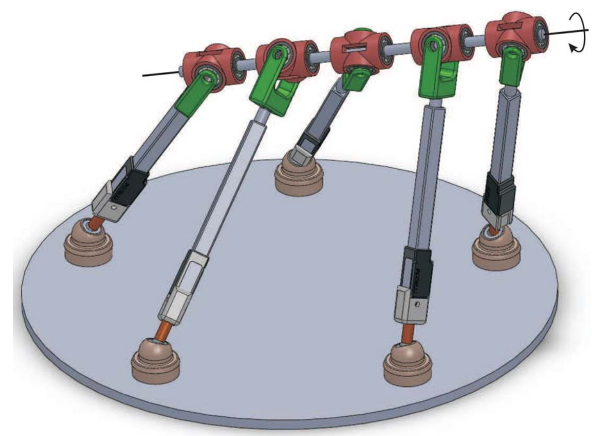

Fig. 1. 5-S$\underline{P}$U parallel robot with aligned universal joints. While the axis defined by these universal joints is rigidly linked to the base for fixed leg lengths, any tool attached to it can freely rotate.

I. Introduction

Over the past half-century, the Stewart–Gough platform has been applied extensively to automate many different tasks due to its well-known advantages in terms of speed, rigidity, dynamic bandwidth, accuracy, cost, etc. [1], [2]. There are many important industrial tasks that require a tool to be perpendicular to a 3-D free-from surface along a given trajectory. They include 5-axis milling, laser engraving, spray-based painting, water-jet cutting, and, in general, any manipulation task in which the tool is axisymmetric. These tasks can be performed by robots with only three translations and two rotations, i.e., five degrees of freedom (DOF). Since the Stewart–Gough platform has 6 DOF, some limited-DOF parallel robots have been designed for this kind of applications with the aim to simplify the structure and the control of the general Stewart–Gough platform but without losing its aforementioned advantages [3]–[7].

The Stewart–Gough platform consists of a base and a moving platform that is connected by six universal-prismatic-spherical (U$\underline{P}$S) legs, where the underline indicates that the prismatic joint is actuated. Thus, it is usually referenced to as a 6-U$\underline{P}$S, or equivalently as a 6-S$\underline{P}$U, parallel mechanism. If one of these legs is eliminated to obtain a 5-DOF parallel robot, two alternatives arise to make the moving platform location controllable, namely, 1) adding an extra passive leg or 2) restraining the mobility of one of the five remaining legs. Then, the challenge consists in how to perform any of these two operations so that the resulting robot has three translations and two rotations. Zhao and colleagues beat the challenge for the first alternative. They proposed to introduce a prismatic-revolute-prismatic-revolute (PRPU) passive leg. The properties of the resulting mechanism, which is technically referenced to as a 5-U$\underline{P}$S + PRPU mechanism for obvious reasons, have been analyzed in a series of papers [8]–[10]. More recently, Lu and colleagues have opted for the second alternative. They proposed a 4-U$\underline{P}$S + S$\underline{P}$R parallel platform whose static and dynamic properties are studied in [11] and [12], respectively. Many other examples of 5-DOF parallel robots can be found in the literature, but they greatly depart from the basic 6-U$\underline{P}$S design in the sense that they do not contain at least four U$\underline{P}$S legs.

A parallel robot that consists of a base and a moving platform that is connected by five S$\underline{P}$U legs is clearly uncontrollable. For example, if the universal joints are aligned as in Fig. 1,

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The paper has supplementary downloadable material provided by the authors, which includes a video of a functioning 5-S$\underline{P}$U prototype, a video illustrating its underlying geometry, and MAPLE worksheets together with webpage explanations for examples of 5-S$\underline{P}$U robots with quartic, cubic, and quadratic forward kinematics solutions, as well as for the redesign example in Section III-D. The material is 10 MB in size.

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the moving platform can freely rotate around the axis that is defined by the five aligned revolute joints. If this rotation axis is made coincident with the symmetry axis of the tool, the uncontrolled motion becomes irrelevant in most cases, and the five leg actuators control the remaining five DOF. Alternatively, this uncontrolled motion can always be eliminated by blocking one of the five aligned revolute joints. The presented analysis is valid irrespective of this choice. Kong and Gosselin refer to the aforementioned arrangement of five SPU legs as a *line-plane component*, since it can always be considered as a rigid subassembly in a standard Stewart–Gough platform [13], [14].

In 1991, Zhang and Song solved, for the first time, the forward kinematics of a general Stewart–Gough platform that contains a line-plane component [15], [16]. They showed how the line in the line-plane component of such a platform can have up to eight configurations with respect to the plane, and as a consequence, the platform can have up to 16 assembly modes. The eight configurations of the line correspond to the roots of a biquartic polynomial. Therefore, the existence of an algebraic expression for these configurations as a function of the five leg lengths was proved. Later on, in 2000, Husty and Karger studied the conditions for this subassembly to be architecturally singular and found two algebraic conditions that must be simultaneously satisfied [17]. More recently, Borràs and Thomas have analyzed the role of cross ratios between the location coordinates of the spherical and universal joints centers—which will be referred to as *attachments* in what follows—in the characterization of architectural singularities, and in singularity-invariant architectural changes, in line-plane components [18].

The parallel singularities of the Stewart–Gough platform have been extensively studied, mainly from an analytic viewpoint [19], [20]. A few works have attained a geometric characterization of the singularity locus for particular platform architectures [21], [22], such as 6-4 platforms [23] and the octahedral manipulator [24], [25]. Similarly, we derive, here, a simple geometric condition that completely characterizes the singularity locus of 5-SPU robots that have a line-plane structure. Moreover, in our search for transformations of robot designs that leave the singularity locus invariant, we introduce the 3-D space of leg rearrangements, which turns out to be a useful tool to characterize all robot instances that have exactly the same locus. Moreover, this space permits us to further group robot instances into families that have topologically equivalent singularity structures. It is proved that there are only three such families, corresponding to robots whose forward kinematics have a quartic, cubic, or quadratic solutions, respectively.

Then, quadratically solvable 5-SPU robots are studied in detail. We show that this family is characterized by a simple algebraic relation between the base and the platform attachment coordinates, which makes the number of possible assembly modes drop to 4 so that they can be computed by solving two quadratic polynomials. In addition, the singularity locus becomes so simple that, for a fixed orientation, it reduces to a plane.

The rest of this paper is organized as follows. Section II presents the kinematics and singularity analysis of the general 5-SPU platform, yielding the eight assembly modes. Next, leg rearrangements that preserve the singularity locus are studied in Section III and then proceed to the classification of 5-SPU platforms according to their singularity structure in Section IV. The family of quadratically solvable robots is studied in detail in Section V, showing that the number of assembly modes drops to 4, and the singularity structure is greatly simplified, as presented in Section V-C. Finally, Section VI points out the implications of the results that are obtained for the study of 6-UPS Stewart–Gough platforms that contain a line-plane component.

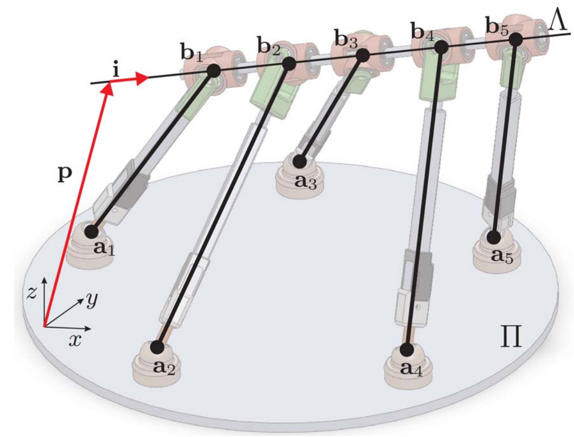

Fig. 2. Schematic representation of the 5-SPU parallel robot that is shown in Fig. 1.

II. 5-SPU Robot With Planar Base and Linear Platform

Let us consider the five-leg parallel platform that is shown in Fig. 2, whose base and platform attachments lie on plane Π and line Λ, respectively. We assume that no four attachments in the base plane are collinear; otherwise, the mechanism would contain a four-leg rigid subassembly, which has been studied separately [26]. Let Π coincide with the xy-plane of the base reference frame. Thus, the leg attachments in the base have coordinates $\mathbf{a}_i = (x_i, y_i, 0)^T$, for $i = 1, \ldots, 5$. The pose of Λ with respect to Π can be described by the position vector $\mathbf{p} = (p_x, p_y, p_z)^T$ and the unit vector $\mathbf{i} = (u, v, w)^T$ in the direction of Λ. Thus, the coordinates of the leg attachments in Λ, which are expressed in the base reference frame, can be written as $\mathbf{b}_i = \mathbf{p} + z_i\mathbf{i}$.

It is worth emphasizing that the attachments of the ith leg can be determined by a single point in $\mathbb{R}^3$ with coordinates (x_i, y_i, z_i). This *3-D space of leg attachments* will play an important role later in Sections III and IV-B.

A. Singularity Analysis

It has previously been shown [18] that the Jacobian determinant of a general Gough-Stewart platform that contains a five-leg line-plane component factorizes into two terms: one that vanishes when the sixth leg lies on the platform plane and the other being the determinant of the following matrix:

$$\mathbf{T} = \begin{pmatrix} wp_z & w(p_zu - p_xw) & w(p_zv - p_yw) & p_z(p_xw - p_zu) & p_z(p_yw - p_zv) & -w^2 \\ z_1 & x_1 & y_1 & x_1z_1 & y_1z_1 & 1 \\ z_2 & x_2 & y_2 & x_2z_2 & y_2z_2 & 1 \\ z_3 & x_3 & y_3 & x_3z_3 & y_3z_3 & 1 \\ z_4 & x_4 & y_4 & x_4z_4 & y_4z_4 & 1 \\ z_5 & x_5 & y_5 & x_5z_5 & y_5z_5 & 1 \end{pmatrix} \quad (1)$$

which depends exclusively on the five-leg 5-DOF component.

Thus, the singularity locus of the 5-SPU manipulator that is studied in this paper corresponds to the root locus of the polynomial that results from expanding such determinant, i.e.,

$$C_1wp_z + C_2w(p_zu - p_xw) + C_3w(p_zv - p_yw) + C_4p_z(p_xw - p_zu) + C_5p_z(p_yw - p_zv) - C_6w^2 = 0 \quad (2)$$

where C_i, for $i = 1, \ldots 6$ is the cofactor of the $(1, i)$ entry of $\mathbf{T}$, which depends only on leg attachments. In what follows, we assume that not all C_i are equal to zero since, in this case, det($\mathbf{T}$) would be identically zero, irrespective of the pose of the platform, which would, thus, be architecturally singular.

B. Forward Kinematics

Similar to [16], the forward kinematics of our five-leg parallel robot can be solved by writing the leg lengths as $l_i = \|\mathbf{b}_i - \mathbf{a}_i\|$, for $i = 1, \ldots, 5$. Then, subtracting from the expression for l_i^2, $i = 1, \ldots, 5$, the equation $\|\mathbf{i}\| = u^2 + v^2 + w^2 = 1$, quadratic terms in u, v, and w cancel out, yielding

$$z_it - x_ip_x - y_ip_y - x_iz_iu - y_iz_iv + \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + x_i^2 + y_i^2 + z_i^2 - l_i^2) = 0 \quad (3)$$

for $i = 1, \ldots, 5$, where $t = \mathbf{p} \cdot \mathbf{i}$.

Subtracting the first equation from the others, quadratic terms in p_x, p_y, and p_z cancel out as well. Then, the resulting system of equations can be written in matrix form as

$$\begin{pmatrix} x_2 - x_1 & y_2 - y_1 & x_2z_2 - x_1z_1 & y_2z_2 - y_1z_1 \\ x_3 - x_1 & y_3 - y_1 & x_3z_3 - x_1z_1 & y_3z_3 - y_1z_1 \\ x_4 - x_1 & y_4 - y_1 & x_4z_4 - x_1z_1 & y_4z_4 - y_1z_1 \\ x_5 - x_1 & y_5 - y_1 & x_5z_5 - x_1z_1 & y_5z_5 - y_1z_1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ u \\ v \end{pmatrix} = \begin{pmatrix} (z_2 - z_1)t + N_2 \\ (z_3 - z_1)t + N_3 \\ (z_4 - z_1)t + N_4 \\ (z_5 - z_1)t + N_5 \end{pmatrix} \quad (4)$$

where

$$N_i = \frac{1}{2}(x_i^2 + y_i^2 + z_i^2 - l_i^2 - x_1^2 - y_1^2 - z_1^2 + l_1^2). \quad (5)$$

Now, notice that the determinant that is associated with the linear system (4) can be written as

$$\begin{vmatrix} x_1 & y_1 & x_1z_1 & y_1z_1 & 1 \\ x_2 & y_2 & x_2z_2 & y_2z_2 & 1 \\ x_3 & y_3 & x_3z_3 & y_3z_3 & 1 \\ x_4 & y_4 & x_4z_4 & y_4z_4 & 1 \\ x_5 & y_5 & x_5z_5 & y_5z_5 & 1 \end{vmatrix} \quad (6)$$

which coincides with C_1 in (2). If (6) vanishes, either p_x, p_y, u, or v can be chosen as parameter, instead of t, to reformulate the linear system (4). Since for a non-architecturally singular robot not all cofactors are zero, it can be shown that a non-singular linear system of the form (4) can always be found by choosing either t, p_x, p_y, u, or v as parameter.

Solving (4) by Cramer's rule, and applying the multilinearity property of determinants, yields

$$\begin{aligned} p_x &= (-C_2t + E_2)/C_1 \\ p_y &= (-C_3t + E_3)/C_1 \\ u &= (-C_4t + E_4)/C_1 \\ v &= (-C_5t + E_5)/C_1 \end{aligned} \quad (7)$$

where E_i results from substituting the $(i-1)$th column vector of the matrix in the system (4) by $(N_2, \ldots, N_5)^T$ and computing its determinant.

From equation $u^2 + v^2 + w^2 = 1$ and (3) for $i = 1$, it can be concluded that

$$\begin{aligned} p_z^2w^2 = {} & (1 - u^2 - v^2) \\ & \times [2(-z_1t + x_1p_x + y_1p_y + z_1y_1v + z_1x_1u) \\ & -p_x^2 - p_y^2 - x_1^2 - y_1^2 - z_1^2] . \end{aligned} \quad (8)$$

On the other hand, from $t = \mathbf{p} \cdot \mathbf{i}$

$$(p_zw)^2 = (t - p_xu - p_yv)^2. \quad (9)$$

Equating the right-hand sides of (8) and (9), the following polynomial in t is finally obtained:

$$n_4t^4 + n_3t^3 + n_2t^2 + n_1t + n_0 = 0 \quad (10)$$

where

$$\begin{aligned} n_4 &= -\frac{(C_4C_3 - C_2C_5)^2}{C_1^4} \\ n_3 &= -\frac{2}{C_1^4}(C_1^2(C_5C_3 + C_4C_2) \\ &\quad + C_1(C_5^2 + C_4^2)(C_2x_1 + (C_1 + C_4x_1 + C_5y_1)z_1 + y_1C_3) \\ &\quad + (C_4C_3 - C_5C_2)(E_5C_2 + E_2C_5 - E_4C_3 - E_3C_4)) \end{aligned} \quad (11)$$

and n_2, n_1, and n_0 also depend on constant parameters but are not provided for space reasons.

Each of the four roots of (10) determines a single value for p_x, p_y, u, and v through (7) and two sets of values for p_z and w by simultaneously solving $\|\mathbf{i}\| = 1$ and $t = \mathbf{p} \cdot \mathbf{i}$. Thus, up to eight assembly modes are obtained for a given set of leg lengths.

The polynomial in (10) is the maximum degree polynomial that we have to solve to obtain the forward kinematics solutions; therefore, we say that the general solution for the 5-S$\underline{\text{P}}$U manipulator with planar base and linear platform is *quartic*.

III. Singularity-Invariant Leg Rearrangements

Now, we want to explore possible changes of leg attachments in both the planar base Π and the linear platform Λ that leave the robot's singularity locus invariant. To this aim, we first interpret the singularity equation (2) as an unfolding of a surface in the 3-D space of leg attachments, whose simple characterization in terms of a distinguished point (which is denoted $\mathcal{B}$ in what follows) and a single line through it (which is denoted $\mathcal{B}_\infty$) permits us to derive geometric rules to perform the sought singularity-invariant leg rearrangements.

A. Algebraic Formulation

Consider the following 2-D surface in $\mathbb{R}^3$:

$$\begin{vmatrix} z & x & y & xz & yz & 1 \\ z_1 & x_1 & y_1 & x_1z_1 & y_1z_1 & 1 \\ z_2 & x_2 & y_2 & x_2z_2 & y_2z_2 & 1 \\ z_3 & x_3 & y_3 & x_3z_3 & y_3z_3 & 1 \\ z_4 & x_4 & y_4 & x_4z_4 & y_4z_4 & 1 \\ z_5 & x_5 & y_5 & x_5z_5 & y_5z_5 & 1 \end{vmatrix} = 0 \tag{12}$$

which can be interpreted as the hypersurface defined by points $(x_i, y_i, z_i), i = 1, \ldots 5$ in the 3-D space of leg attachments that are introduced in the preceding section. The Laplace expansion by the elements of the first row of such determinant leads to

$$C_1z + C_2x + C_3y + C_4xz + C_5yz + C_6 = 0 \tag{13}$$

where C_i are the cofactors of the elements of the first row, for $i = 1, \ldots, 5$. Note that these are the same coefficients as those in the singularity polynomial (2). If any leg is substituted by a new one going from the base attachment $\mathbf{a} = (x, y, 0)$ to the platform attachment $\mathbf{b} = \mathbf{p} + z\mathbf{i}$, for any (x, y, z) that satisfy (13), the values of the coefficients C_i for $i = 1, \ldots, 6$ will remain the same up to a constant multiple. Hence, the points with the coordinates of the five leg attachments belong to the surface that is defined by (13), and we can freely move them within this surface without altering the platform's singularity locus. This is because the coefficients of the singularity polynomial in (2) remain the same up to a scalar multiple and, as a consequence, its root locus remains invariant. The only caution required is that this scalar multiple be different from zero, as otherwise the platform would be architecturally singular. It is worth noting that, in this case, the coordinates of the resulting five legs would not define a surface in implicit form through (12).

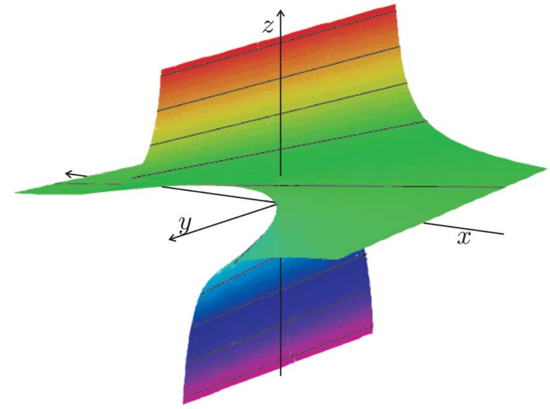

Fig. 3. Representation of surface (13) with the origin placed at point (16) and the y-axis that is placed at line (15).

B. Geometric Rules to Perform Leg Rearrangements

We like to study what leg rearrangements leave the surface that is defined by (13) unchanged and, thus, keep the platform singularity locus invariant. To this aim, let us rewrite (13) in matrix form as

$$[(C_2\ C_3\ C_6) + z(C_4\ C_5\ C_1)] \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{14}$$

For each pair (x, y), there is a unique corresponding z through (14), provided $(C_4x + C_5y + C_1) \neq 0$. Conversely, for each value of z, (14) defines a unique line in variables x and y. This also holds for $z = \infty$, whose corresponding line is

$$\mathcal{B}_\infty = \{(x, y) \mid C_4x + C_5y + C_1 = 0\}. \tag{15}$$

Equation (14) has the form of a projective pencil of lines, where each line of the pencil is formed by a linear combination of the line (15) and the line $C_2x + C_3y + C_6 = 0$. Then, the vertex of the pencil is the point that belong to both lines, i.e.

$$\mathcal{B} = \left(\frac{C_3C_1 - C_6C_5}{C_2C_5 - C_4C_3}, -\frac{C_2C_1 - C_4C_6}{C_2C_5 - C_4C_3} \right) \tag{16}$$

for which any value of z satisfies (14).

Fig. 3 shows that the surface that is defined by (13) has the shape of a spiral-like ruled surface around a vertical axis that passes through point $\mathcal{B}$ (16) in the xy-plane and approaching a line that is parallel to (15) as z tends to ∞. This can be recognized as a hyperbolic paraboloid with two directing lines at infinity, which are obtained by intersecting the planes $z = 0$ and $C_4x + C_5y + C_1 = 0$ with the plane at infinity.

Interpreting this surface in the 3-D space of leg attachments—where (x, y) and z are the coordinates of the attachments in the base plane Π and the platform line Λ, respectively—we note that (14) defines a one-to-one correspondence between points in Λ and lines of a pencil in Π, with vertex at $\mathcal{B}$ (see Fig. 4).

In what follows, any line in Π that passes through point $\mathcal{B}$ will be called a $\mathcal{B}$-*line*. The $\mathcal{B}$-line that is associated with the attachment in Λ with local coordinate z_i will be denoted $\mathcal{B}_{z_i}$. Of particular interest is $\mathcal{B}_\infty$, which that is given in (15), because in practice, no attachment in Π can be located on it (with the

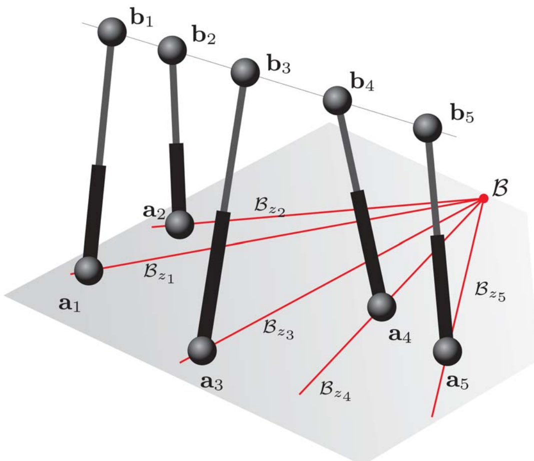

Fig. 4. One-to-one correspondence between the attachments in the platform line and the lines of the pencil centered at $\mathcal{B}$. Each value of z_i defines a point in the platform line $\mathbf{b}_i = \mathbf{p} + z_i\mathbf{i}$ and a line in the plane $\mathcal{B}_{z_i}$.

exception of $\mathcal{B}$), as the corresponding attachment on Λ should have to be moved to infinity. Moreover, the surface that is defined by (13) will be called *$\mathcal{B}$-surface* when interpreted in the 3-D space of leg attachments.

Summarizing, we can state two simple rules to move the leg attachments without altering the singularity locus of a given 5-$S\underline{P}U$ platform with planar base and linear platform as follows:

1) For fixed platform attachments, all attachments in the base plane can be freely moved along their $\mathcal{B}$-lines.
2) For fixed base attachments, an attachment in the linear platform can be freely moved if, and only if, the corresponding attachment in the base is located at $\mathcal{B}$.

Again, the only caution required is to avoid falling into architecturally singular designs, which can be easily detected because all C_i's, $i = 1 \ldots 6$, would be zero. These architecturally singular designs that are originated by degeneracies, such as placing three attachments on the same $\mathcal{B}$-line or having four collinear attachments on the base, were already characterized in [18], [29].

C. Geometric Interpretation of Parallel Singularities

Let us rewrite (2) in vector form as

$$[w(C_2\ C_3\ C_6) - p_z(C_4\ C_5\ C_1)]\begin{pmatrix} p_x w - p_z u \\ p_y w - p_z v \\ w \end{pmatrix} = 0. \quad (17)$$

The parallel singularities of the analyzed 5-S$\underline{P}$S robot correspond to those configurations, which are defined by $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{i} = (u, v, w)$, that satisfy the aforementioned equation. Then, two situations arise.

1) If $w \neq 0$, (17) yields

$$[(C_2\ C_3\ C_6) + \mu(C_4\ C_5\ C_1)]\begin{pmatrix} p_x + \mu u \\ p_y + \mu v \\ 1 \end{pmatrix} = 0 \quad (18)$$

where $\mu = -p_z/w$. The first term of the equation defines a pencil of lines, which is the same pencil obtained in the previous section. Now, observe that Λ intersects Π at

$$\mathcal{A} = (p_x + \mu u, p_y + \mu v, 0). \quad (19)$$

Then, according to (18), the singularity occurs when point $\mathcal{A}$ lies on the line that is defined by $\mathcal{B}_0 + \mu\mathcal{B}_\infty$, i.e., the line of the pencil corresponding to $z = -p_z/w$. Note that, if $\mathcal{A}$ coincides with $\mathcal{B}$, i.e., the focus of the pencil, the manipulator would be singular for any value of p_z and w, because $\mathcal{A}$ would simultaneously lay on all lines of the pencil.

2) If $w = 0$, (17) yields

$$(C_4\ C_5)\begin{pmatrix} p_z u \\ p_z v \end{pmatrix} = 0. \quad (20)$$

In this case, the manipulator is singular when Λ is parallel to $\mathcal{B}_\infty$, i.e., when $\mathbf{i} = \pm\frac{1}{\sqrt{C_4^2 + C_5^2}}(C_5, -C_4, 0)$.

If, in addition, $p_z = 0$, Λ necessarily lies on Π, which is a trivial singularity.

In sum, the 5-S$\underline{P}$U manipulator is in a singular configuration iff the platform point $\mathbf{p} + z\mathbf{i}$ that intersects the base does so precisely at its corresponding $\mathcal{B}$-line $\mathcal{B}_z$ (see Fig. 5 for an example). Note that this includes the cases in which $w = 0$.

The aforementioned geometric interpretation has two very interesting implications. First, a configuration is singular iff a leg can attain zero length through a singularity-invariant leg rearrangement. The attachments of such a leg will both coincide with the point where the platform intersects the base. Second, this zero-length leg condition that holds at singularities permits us to equate the coordinates of attachments in the base $\mathbf{a} = (x, y, 0)^T$ and platform $\mathbf{b} = \mathbf{p} + z\mathbf{i}$ at point $\mathcal{A}$, leading to the following change of variables:

$$\begin{aligned} xw &= p_x w - p_z u \\ yw &= p_y w - p_z v \\ zw &= p_z \end{aligned} \quad (21)$$

which, if applied to (2), yields

$$(-w^2)(C_1 z + C_2 x + C_3 y + C_4 zx + C_5 zy + C_6) = 0. \quad (22)$$

When $w \neq 0$, this reduces to (13). Therefore, except for configurations in which the platform lies parallel to the base, the $\mathcal{B}$-surface (13) in the 3-D space of leg attachments provides a characterization of singularities that are equivalent to the hypersurface (2) in the 5-D robot configuration space.

D. Example I

Multiple spherical joints exist in most well-studied Gough–Stewart platforms. Such joints simplify the kinematics and singularity analysis of parallel manipulators, but they are difficult to construct and present small joint ranges, which make them of little practical interest. In this example, it is shown how the presented leg rearrangements can be used to eliminate multiple spherical joints from a particular design, without losing the advantages of having simple kinematics and maintaining the same singularity locus.

Consider the 5-S$\underline{P}$U manipulator that is depicted in Fig. 6 (top), which is clearly of the line-plane type that is studied in

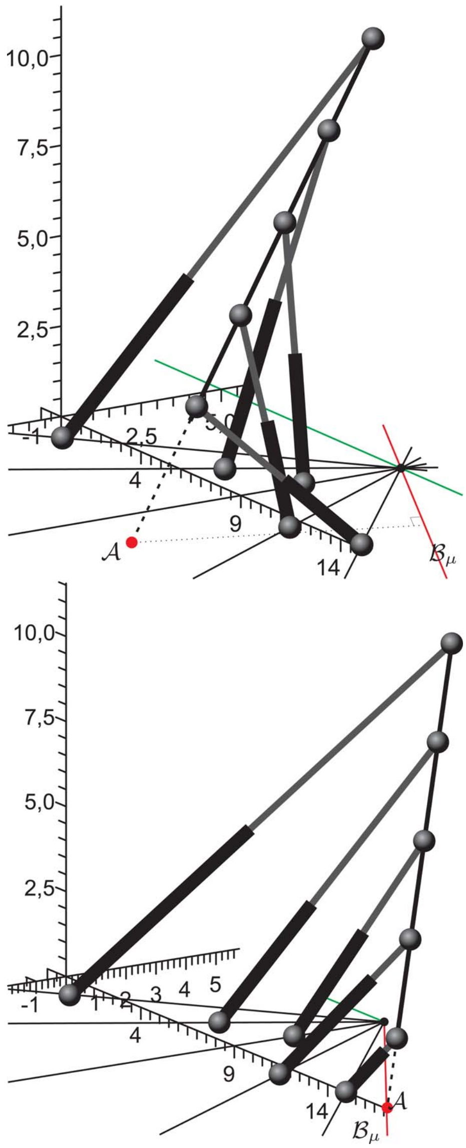

Fig. 5. Non-singular pose of the manipulator, for the position $\mathbf{p} = (5, 8, 13)$, $\mathbf{i} = (1/3, -2/3, -2/3)$ (top). A singular pose, $\mathbf{p} = (7\sqrt{6} - 7, 4, 14)$ and $\mathbf{i} = (\frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{3})$ of the manipulator (bottom).

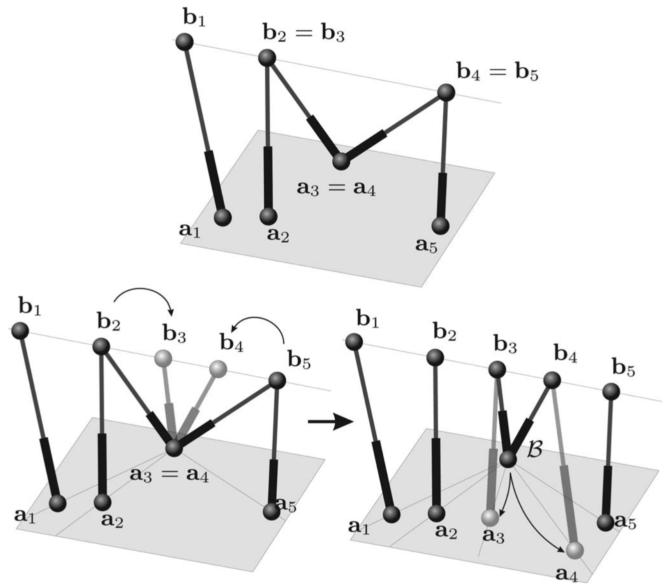

Fig. 6. Singularity-invariant leg rearrangements can be used at the manipulator design stage to eliminate multiple spherical joints.

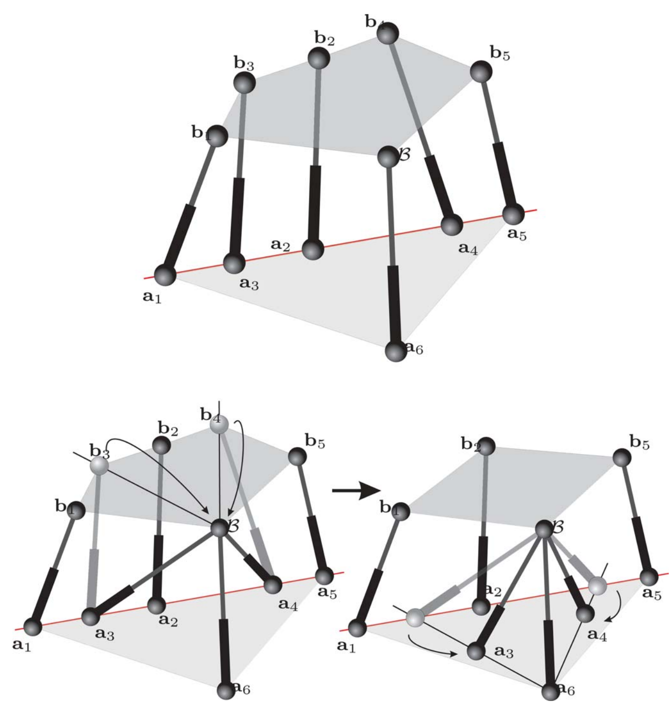

Fig. 7. From a plane-line component in the top figure, an uncoupled manipulator is obtained using singularity-invariant leg rearrangements.

this paper. A set of leg rearrangements can be performed to transform it into a platform with the same singularities but with no multiple spherical joints. One of the possible sequences of leg rearrangements to attain this goal appears in Fig. 6 (bottom).[1]

Two remarks may ease the practical application of the leg rearrangement rules that are presented in the preceding section.

1) There can be at most two coincident attachments on the base plane, which must lie on point $\mathcal{B}$. Otherwise, the manipulator either would contain a four-leg rigid component or it would be architecturally singular.
2) Along a design process, the location of point $\mathcal{B}$ may be conveniently specified by placing two coincident attachments, which can be separated later on using appropriate leg rearrangements.

E. Example II

Consider the Stewart–Gough platform shown in Fig. 7 (top). It contains an upside-down line-plane component. Hence, the associated pencil of lines lies, in this case, in the platform plane.

[1]Check file 04_The_3-4_5-UPS.mw in the multimedia attached archive for a numerical example.

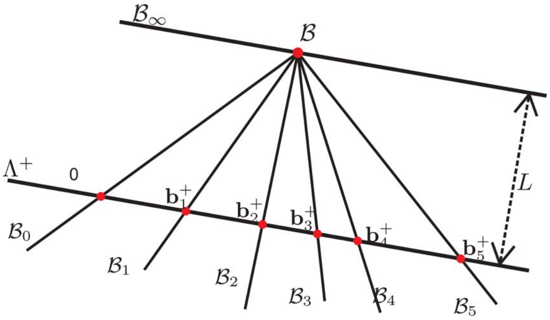

Fig. 8. Planar geometric construction that defines all the geometric parameters in a 5-SP̲U manipulator with planar base and linear platform.

Moreover, the attachment in the platform of the leg that is not included in the line-plane component is made to be coincident with the focus of the pencil $\mathcal{B}$.

According to the results that are presented in Section III-B, two platform attachments can be moved along their $\mathcal{B}$-lines to meet at $\mathcal{B}$ without modifying the singularity locus of the considered platform. A point-plane component, thus, arises [see Fig. 7 (left-bottom)]. It can be shown that the attachments in the plane of a point-plane component can be arbitrarily relocated, without changing the singularity locus of the whole platform, provided that no architectural singularities are introduced [27]. As a consequence, it is possible to misalign two of the base attachments [see Fig. 7 (right-bottom)]. The result is an uncoupled parallel platform because the legs of the point-plane component determine the location of a point in the moving platform and the other three legs, the platform's orientation. It can be said that the resulting uncoupled manipulator contains a concealed line-plane component. Thus, it is clear that the presented study transcends that of 5-SP̲U platforms.

IV. Classifying 5-SP̲U Platforms by Their Singularities

A. Platform Families With Identical Singularities

Once the leg rearrangements that preserve singularity loci have been identified, we like to classify platforms in families that share each such locus. To this end, we first identify the geometric entities that fully describe the singularity locus.

It is interesting to realize that it is possible to locate a copy of Λ onto Π, parallel to the line $\mathcal{B}_\infty$

$$\Lambda^+ = \left\{ (x,y) |\, C_4 x + C_5 y + C_1 + \frac{C_2 C_5 - C_3 C_4}{\sqrt{C_4^2 + C_5^2}} = 0 \right\} \tag{23}$$

so that each attachment in Λ^+ lies on its associated $\mathcal{B}$-line in Π (see Fig. 8).

Let us denote the coordinates of the intersections of Λ^+ with $\mathcal{B}_{z_i}$ by $\mathbf{b}_i^+$. Notice that $\mathbf{b}_i^+$, $i = 1, \ldots, 5$, are spaced at the same distances in Λ^+ as $\mathbf{b}_i$, $i = 1, \ldots, 5$, in Λ. Then, Λ^+ is a privileged line in Π that represents a possible location for Λ so that the attachments in it coincide with their corresponding $\mathcal{B}$-lines.

Given a particular manipulator, point $\mathcal{B}$, line $\mathcal{B}_\infty$, and line Λ^+ can be computed using (16), (15), and (23), respectively. These determine the five $\mathcal{B}$-lines passing through the base attachments, and their intersections with Λ^+, $\mathbf{b}_i^+$, $i = 1, \ldots, 5$, determine also the location of the attachments $\mathbf{b}_i$, $i = 1, \ldots, 5$ in Λ (see Fig. 8).

As a consequence, point $\mathcal{B}$, line $\mathcal{B}_\infty$, and line Λ^+ characterize a family of 5-SP̲U manipulators having exactly the same singularity locus. Furthermore, assuming that point $\mathcal{B}$ is finite, we can always apply a planar affine transformation that moves $\mathcal{B}$ to the origin and line B_∞ to the y-axis. Then, the $\mathcal{B}$-surfaces associated with two non-architecturally singular 5-SP̲U manipulators differ at most on a *scaling factor*, namely the distance of point $\mathcal{B}$ to line Λ^+ (which is named L in Fig. 8). This factor regulates the attachments spacing in the platform line in relation to the attachments spacing in the base plane.[2]

Therefore, all non-architecturally singular 5-SP̲U manipulators with a finite point $\mathcal{B}$ have associated $\mathcal{B}$-surfaces with the same topology. Moreover, through the change of variables in (21), we can conclude that the singularity loci of all these manipulators also have the same topology.

B. Three Possible Topologies for the Singularity Locus

So far, we have assumed that point $\mathcal{B}$ was finite. Now, suppose we take it to infinity. According to (16), this implies that $C_2 C_5 - C_4 C_3 = 0$. By introducing this constraint into (13), we obtain

$$(C_4 z + C_2)x + (C_3/C_2)(C_4 z + C_2)y + C_1 z + C_6 = 0. \tag{24}$$

It turns out that all $\mathcal{B}$-lines have now the same slope, $C_3/C_2 = C_5/C_4$, and, therefore, they are all parallel to $\mathcal{B}_\infty$. Fig. 9 (center) shows the corresponding $\mathcal{B}$-surface with the y-axis that is placed at line $\mathcal{B}_\infty$. Note, thus, that the $\mathcal{B}$-surface approaches asymptotically line $\mathcal{B}_\infty$ as z tends to $+/-\infty$. Moreover, the $\mathcal{B}$-line associated with the value of z for which $C_4 z + C_2 = 0$ is the line at infinity. This appears as the surface asymptotically approaches a horizontal plane $C_4 z + C_2 = 0$ in the central graphic in Fig. 9, which can be recognized as a hyperbolic cylinder.

Thus, it is worth remarking that, in the one-to-one correspondence between points in Λ and lines in Π, we have here that a finite point in Λ has its associated $\mathcal{B}$-line at infinity, while the point at infinity in Λ is associated with the finite $\mathcal{B}_\infty$ line.

Next, let us explore what would happen if these two lines are made to be coincident, i.e., $\mathcal{B}_\infty$ is taken to infinity. Since point $\mathcal{B} \in \mathcal{B}_\infty$, $\mathcal{B}$ also stays at infinity as earlier. This further condition implies that $C_4 = C_5 = 0$, and (24) reduces to

$$C_2 x + C_3 y + C_1 z + C_6 = 0. \tag{25}$$

Of course, all $\mathcal{B}$-lines continue to be parallel, but observe that their spacing has now become a linear function of z, namely, $C_1 z + C_6$. Thus, the $\mathcal{B}$-surface is a plane in this case. Fig. 9 (right) shows this planar $\mathcal{B}$-surface with $\mathcal{B}$-lines parallel to the

[2] To visualize the effect of moving line Λ^+ and point $\mathcal{B}$ on the geometry of the manipulator, a video has been attached as multimedia material definingGeometricElements.avi

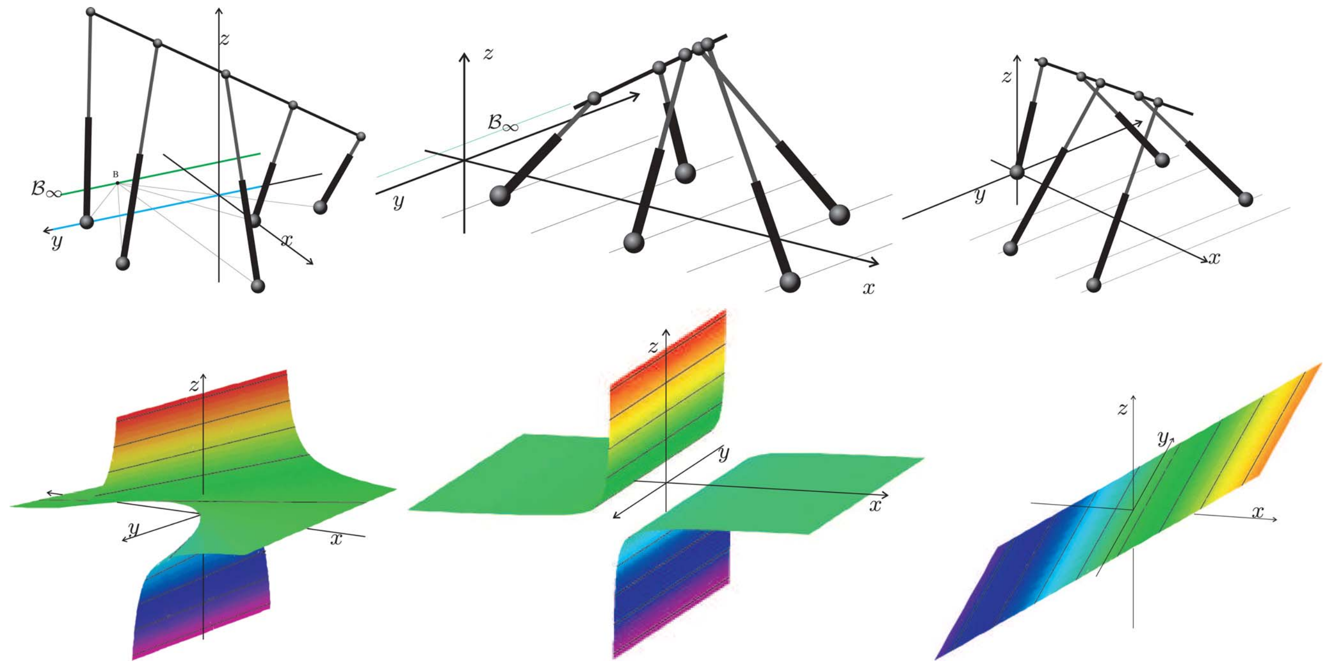

Fig. 9. Quartic, cubic, and quadratically solvable 5-SP̲U manipulators (from top-left to top-right) with their corresponding $\mathcal{B}$-surfaces (bottom).

y-axis. Note that the $\mathcal{B}$-surface approaches line $\mathcal{B}_\infty$ linearly as z tends to $+/-\infty$.

In sum, there are only three possible topologies for the $\mathcal{B}$-surfaces associated with non-architecturally singular 5-SP̲U manipulators: one when point $\mathcal{B}$ is finite [see Fig. 9 (left)], another when $\mathcal{B}$ is taken to infinity but $\mathcal{B}_\infty$ remains finite [see Fig. 9 (center)], and the third when both point $\mathcal{B}$ and line $\mathcal{B}_\infty$ are taken to infinity [see Fig. 9 (right)]. Again, through the change of variables in (21), we can conclude that the manipulators in each of these three families have singularity loci with the same topology.

C. Quartic, Cubic, and Quadratic Cases

At the end of Section II-B, we mentioned that the general solution of the forward kinematics for the 5-SP̲U manipulator with planar base and linear platform is *quartic*, since it entails finding the roots of polynomial (10).

Now note that, when point $\mathcal{B}$ lies at infinity, $C_2C_5 - C_4C_3 = 0$, the leading coefficient n_4 in (10) vanishes, and the forward kinematic solution becomes *cubic*. Then, we only obtain six assembly modes for the platform line Λ. Finally, if not only $\mathcal{B}$ is at infinity, but also line $\mathcal{B}_\infty$ (i.e., $C_4 = C_5 = 0$), it is easy to see that the coefficient n_3 in (10) also becomes zero, leading to a *quadratic* solution. When this happens, the maximum simplification of the kinematics is obtained: a platform with four assembly modes.

Thus, let us remark that the three topologies of the singularity locus derived in the preceding section correspond to the quartically solvable 5-SP̲U robot family, the cubically solvable family, and the quadratically solvable one (see Fig. 9).

D. Singularity Hypersurface Analysis

Let us briefly discuss what the slices of the singularity hypersurface for a fixed platform orientation would look like for each topology.

For the quartic case, taking $(u, v, w) = (0, 0, -1)$, which corresponds to the platform line Λ that is placed perpendicular to the base plane Π, the 2-D slice will look exactly as the $\mathcal{B}$-surface that is displayed in Fig. 3, since (2) reduces to (13). Thus, every neighboring point in the sphere of platform orientations will correspond a slightly different 2-D slice, and we can visualize the 4-D singularity hypersurface as the combination of these spherically arranged 2-D slices.

Fig. 10 (top) illustrates the evolution of the singularity slice when (u, v, w) moves from one pole (0,0,1) toward the equator $(u, v, 0)$ of the sphere of orientations. In general, the spiral-like surface progressively flattens and, for the limiting case in which $w = 0$, it becomes a plane. Note that this relates to the assumption $w \neq 0$ that we made in the change of variables (21). When $w = 0$, the platform line Λ of the manipulator is parallel to the base plane Π, and the equation of the singularity locus reduces to $p_z^2(C_4u + C_5v) = 0$. Two subcases need to be distinguished: $p_z = 0$ and $C_4u + C_5v = 0$. In the former, Λ lies on Π, and the spiral surface becomes the plane p_xp_y, as previously mentioned. In the latter subcase, i.e., when $C_4u + C_5v = 0$, Λ is parallel to the B_∞-line, and the singularity slice at these two equator points covers the whole space $\mathbb{R}^3$ of coordinates p_x, p_y, p_z.

In sum, the singularity locus of a 5-SP̲U manipulator with a finite point $\mathcal{B}$ is a 4-D hypersurface in $S^2 \times \mathbb{R}^3$ that can be parametrized by coordinates $(u, v, p_x, p_y) \in S^2 \times \mathbb{R}^2$, except at the great circle of S^2 projecting to the $\mathcal{B}_\infty$-line, where p_z can take any value.

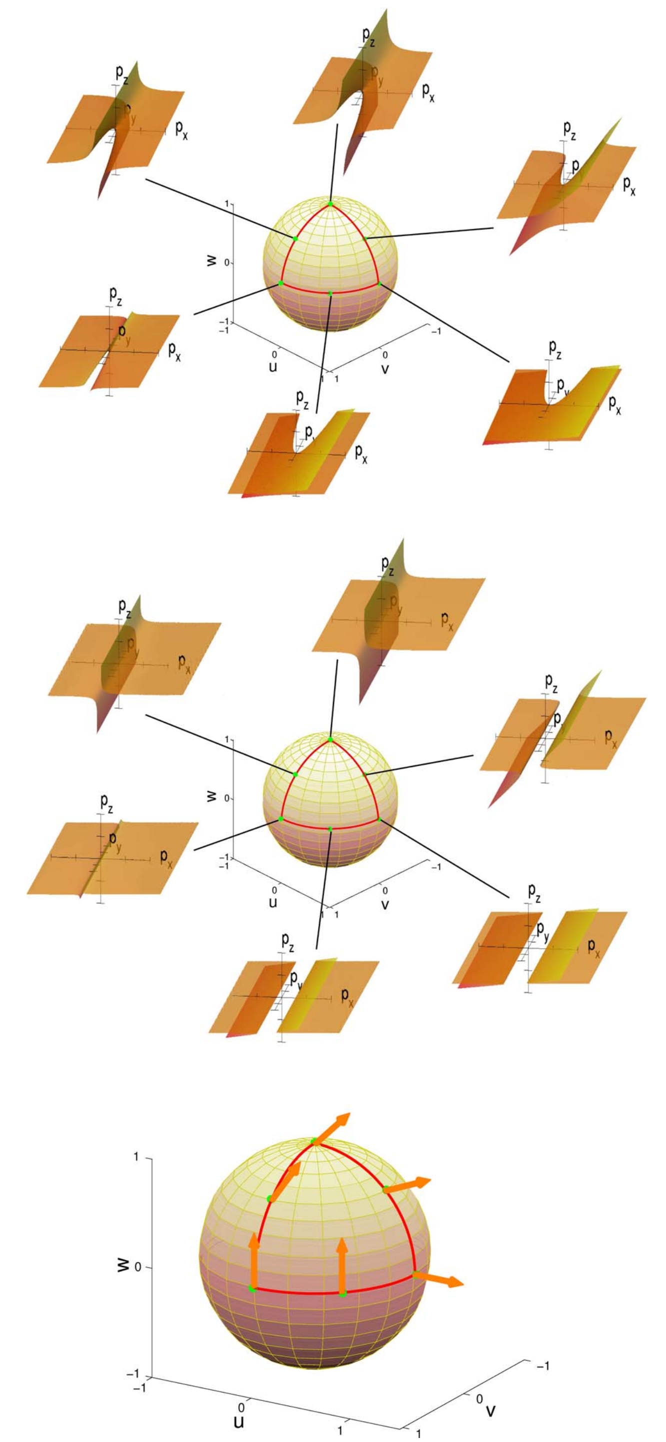

Fig. 10. Evolution of the singularity loci in the space of platform positions $(p_x, p_y, p_z) \in \mathbb{R}^3$ as the platform orientation varies in the sphere $(u, v, w) \in S^2$. (Top) Case in which point $\mathcal{B}$ is at the origin and line $\mathcal{B}_\infty$ coincides with the p_y-axis. (Center) Case in which point $\mathcal{B}$ is at infinity and line $\mathcal{B}_\infty$ coincides with the p_y-axis. Finally, in the case that line $\mathcal{B}_\infty$ is at infinity, the slice of the singularity locus for a each particular orientation is a plane. (Bottom) Normal vector to this plane.

For the two cases with $\mathcal{B}$ at infinity, by fixing as before $(u, v, w) = (0, 0, -1)$, the 2-D slices obtained will look exactly as the corresponding $\mathcal{B}$-surfaces. Then, by making (u, v, w) sweep the sphere of platform orientations, we can visualize each of the three 4-D singularity hypersurfaces as the composition of the spherically arranged 2-D slices. This is done in Fig. 10 (center) for the second topology (cubic case) and in Fig. 10 (bottom) for the third topology (quadratic case).

V. 5-SPU Quadratically Solvable Manipulator

A 5-DOF manipulator whose forward kinematics has a quadratic solution is of interest by itself and as a component to be included in a general 6-DOF Stewart–Gough platform. Hence, we analyze it thoroughly in this section.

Let us consider a quadratically solvable manipulator whose line $\mathcal{B}_\infty$ coincides with the p_y-axis, and thus, its $\mathcal{B}$-lines are parallel to this axis. This implies that we can freely fix its leg attachment coordinates $\mathbf{a}_i = (x_i, y_i, 0)$ and $\mathbf{b}_i = \mathbf{p} + z_i \mathbf{i}$, with $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{i} = (u, v, w)$ as before, subject to the only constraint

$$z_i = \delta x_i \tag{26}$$

where δ is, thus, a proportionality factor between platform attachments and the x-coordinates of the base attachments. To ease readability of the equations, we set $x_1 = y_1 = 0$ without losing generality. Then, δ, x_i, and y_i, $i = 2, 3, 4, 5$, are left as parameters that characterize the family of 5-SPU robots that are analyzed in this section.

A. Forward Kinematics

With the attachment coordinates that are given in (26), the cofactors of the elements of the first row of $\mathbf{T}$ are

$$\begin{aligned} C_1 &= \delta^2 F \\ C_2 &= -\delta^3 F \\ C_3 &= C_4 = C_5 = C_6 = 0 \end{aligned} \tag{27}$$

where F can be written as

$$F = \begin{vmatrix} x_2^2 & x_2 y_2 & x_2 & y_2 \\ x_3^2 & x_3 y_3 & x_3 & y_3 \\ x_4^2 & x_4 y_4 & x_4 & y_4 \\ x_5^2 & x_5 y_5 & x_5 & y_5 \end{vmatrix} \tag{28}$$

and the coefficients of polynomial (10) are

$$\begin{aligned} n_4 &= n_3 = 0 \\ n_2 &= \frac{(\delta^2+1)\delta^2 F^2 - 2\delta F E_4 - E_5^2}{\delta^2 F^2} \\ n_1 &= 2\frac{E_2 \delta^4 F^2 - F\delta(E_4 E_2 + E_5 E_3) - E_5(E_2 E_5 - E_3 E_4)}{\delta^5 F^3} \\ n_0 &= \frac{(E_2^2 + E_3^2 + l_1^2(E_4^2 + E_5^2))F^2 \delta^4 - (E_2 E_5 - E_4 E_3)^2}{\delta^8 F^4} - l_1^2. \end{aligned}$$

Then, polynomial (10) becomes quadratic, and as a consequence, its two roots can be simply expressed as

$$\begin{aligned} t = &\frac{1}{\delta^3 F(2\delta F E_4 + E_5^2 - (\delta^2+1)\delta^2 F^2)} \\ &\cdot [\delta^4 F^2 E_2 - \delta F(E_2 E_4 + E_5 E_3) \\ &+ E_5(E_3 E_4 - E_2 E_5) \pm \sqrt{\Delta}] \end{aligned} \tag{29}$$

where the discriminant is

$$\Delta = \delta F(E_5^2 + E_4^2 - \delta^4 F^2) \times [2\delta^4 F^2 E_4 l_1^2 + \delta^3 F(E_5^2 l_1^2 + E_3^2) + \delta F(E_2^2 + E_3^2) - (\delta^2+1)\delta^5 F^3 l_1^2 + 2E_3(E_2E_5 - E_4E_3)]. \quad (30)$$

Each of the two aforementioned roots, say t_1 and t_2, determines a single value for p_x, p_y, u, and v through (7) and two sets of values for p_z and w by simultaneously solving $\|\mathbf{i}\| = 1$ and $t = \mathbf{p}\cdot\mathbf{i}$. The resulting four assembly modes are explicitly given by

$$\mathbf{p} = \begin{pmatrix} \dfrac{\delta^3 F t_i + E_2}{\delta^2 F} \\ \dfrac{E_3}{\delta^2 F} \\ \pm\dfrac{(E_4 - \delta F)\delta^3 F t_i + E_4E_2 + E_5E_3}{\delta^2 F\sqrt{\delta^4F^2 - E_5^2 - E_4^2}} \end{pmatrix} \quad (31)$$

and

$$\mathbf{i} = \begin{pmatrix} \dfrac{E_4}{\delta^2 F} \\ \dfrac{E_5}{\delta^2 F} \\ \pm\dfrac{\sqrt{\delta^4F^2 - E_5^2 - E_4^2}}{\delta^2 F} \end{pmatrix}. \quad (32)$$

B. Singularity Analysis

Substituting the values of the cofactors (27) into (2), the singular configurations of the studied 5-S$\underline{\text{P}}$U platform are the solutions of the following equation:

$$\delta^2 w F\,[\delta p_x w - (u\delta - 1)p_z] = 0. \quad (33)$$

Observe that, except for δ, all other design parameters are embedded in F, whereas the robot pose appears only in the remaining two factors. Thus, if $F = 0$, the manipulator is architecturally singular, i.e., it is always singular independent of its leg lengths.

Let us now turn to the case $F \neq 0$ and study the parallel singularities of non-architecturally singular manipulators.

A singular configuration $(\mathbf{p}, \mathbf{i}) \in \mathbb{R}^3 \times S^2$, with $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{i} = (u, v, w)$, is that satisfying either $w = 0$ or $[\delta w p_x - (\delta u - 1)p_z] = 0$.

Following the geometric interpretation that is given in Section III-C, when $w = 0$, the manipulator is always in a singularity, because the line Λ is always parallel to the $\mathcal{B}_\infty$ (any line is parallel to a line at infinity, and for the quadratic case, $\mathcal{B}_\infty$ is at infinity). This last condition holds for configurations where the platform is parallel to the base plane.

On the other hand, when $w \neq 0$, (18) reads as

$$[(C_2\ 0\ 0) + \mu(0\ 0\ C_1)]\begin{pmatrix} p_x + \mu u \\ p_y + \mu v \\ 1 \end{pmatrix} = 0$$

where $\mu = -p_z/w$. This condition holds when the intersection point of Λ with Π, which is defined as $\mathcal{A}$ in (19), belongs to the line $C_2 x + \mu C_1 = 0$. In other words, when the point $\mathcal{A}$ is at a distance $(p_z/w)(C_1/C_2) = -(p_z/w\delta)$ from the y-axis, the manipulator is in a singularity.

Note that singularities can also be expressed in joint space $\mathbb{R}^5$ by using the discriminant (30), whose expression only depends on the leg lengths l_i, $i = 1, \ldots, 5$. When $\Delta = 0$, the two solutions (29) coincide, yielding a singularity. Note that Δ also consists of two factors: The first one $E_5^2 + E_4^2 - \delta^4 F^2 = 0$ corresponds to the condition $w = 0$, and the other is equivalent to $(\delta w p_x - (\delta u - 1)p_z) = 0$.

An interesting practical consideration is that, if we fix the orientation of the tool, singularities define a plane in position space [as shown in Fig. 10 (bottom)]

$$c_1 p_x + c_2 p_z = 0 \quad (34)$$

with $c_1 = \delta w^2$ and $c_2 = w(1 - u\delta)$. For example, if the tool is orthogonal to the base plane, i.e., $(u, v, w) = (0, 0, 1)$, then the robot will reach a singularity when its position, i.e., (p_x, p_y, p_z), satisfies

$$\delta p_x + p_z = 0. \quad (35)$$

It follows from the aforementioned singularity analysis that, for a fixed value of δ, the whole family of non-architecturally singular 5-S$\underline{\text{P}}$U robots considered have exactly the same singularity locus. In other words, given a member of the family, one can freely move its leg attachments without modifying the singularity locus, provided two constraints are maintained, namely, the proportionality between x_i and z_i, and $F \neq 0$ in (28) precluding architecturally singular designs.

C. Structure of Configuration Space

The singularity locus of the 5-S$\underline{\text{P}}$U robots studied consists of two hypersurfaces in $\mathbb{R}^3 \times S^2$—the robot configuration space (or C-space, for short)—namely

$$w = 0 \quad \text{and} \quad w p_x - \left(u - \frac{1}{\delta}\right) p_z = 0. \quad (36)$$

Note that since p_y and v do not appear in the hypersurface equations, they do not need to be taken into account when analyzing the topology of singularities. C-space can, thus, be schematically represented by drawing the sphere of orientations in each point of the plane $p_x p_z$. Furthermore, only the projection of the sphere in the direction of the v-axis needs to be displayed. Fig. 11 shows such representation for eight positions around the origin in the plane $p_x p_z$, for the case $\delta = 1$ (the cases $\delta < 1$ and $\delta > 1$ follow easily from this one, as detailed in [28]). Observe that only the relation p_z/p_x is relevant; therefore, each disk stands for all positions in the half-line starting at the origin and having the same p_z/p_x value. Color encodes where the region lies in relation to the two hypersurfaces. For example, yellow points (the brightest gray level ones) are those where $w < 0$ and $w p_x - (u - 1/\delta)p_z < 0$. Lines that separate two colors correspond to the two hypersurfaces.

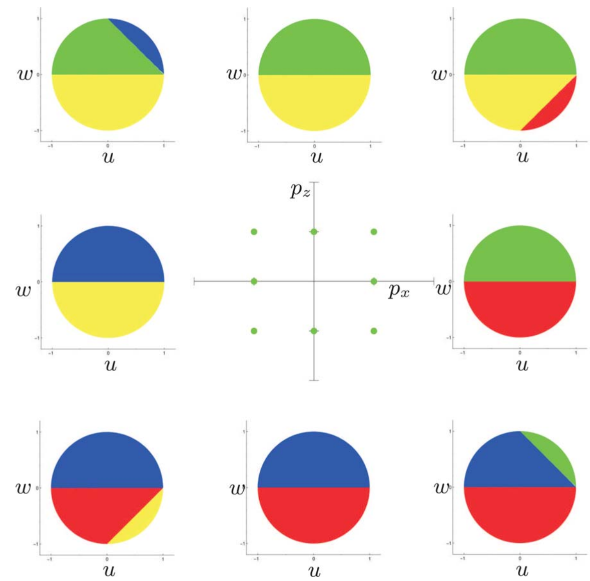

Fig. 11. Sphere of orientations for eight positions around the origin. The four connected components are marked with different colors.

Hence, the two singular hypersurfaces divide C-space into four connected components, corresponding to the four assembly modes in (31) and (32). Note that the symmetry in these equations shows up neatly in the figure. It is worth mentioning that for platform positions in the first quadrant, namely, where $p_x > 0$ and $p_z > 0$, all of the hemisphere of orientations with $w > 0$ is reachable. Similarly, there is a whole hemisphere reachable in the other quadrants.

Further details on the structure of C-space and its cell decomposition induced by the singularity hypersurfaces can be found in [28].

VI. Conclusion

The complete charting of the singular configurations of individual parallel robots is important for motion planning and trajectory control. To obtain rules to perform leg rearrangements that leave the singularity locus unchanged has a more generic interest in that it permits us to optimize robot designs within a repertoire of them without having to care about collateral variations in their singularities. Even further, the establishment of entire robot families with topologically equivalent singularity structures permits us to have a global view of the design options that are available and their associated kinematic complexities.

This paper has presented contributions at these three levels for the case of 5-SPU robots with planar base and linear platform, excluding only non-generic designs such as those with four collinear attachments in the base [26] and architecturally singular ones. It has been shown that there are only three families with distinct topologies for the singularity locus, corresponding to quartically, cubically, and quadratically solvable robot platforms.

The presented analysis of 5-SPU robots is also useful for the study of 6-UPS Stewart–Gough platforms that contain a line-plane component, as has been shown for the decoupled manipulator with three collinear attachments in Section III-E. If such component is of the quadratically solvable type, the kinematics of the 6-DOF platform becomes greatly simplified, having a total of eight assembly modes. A cell decomposition of its singularity locus can be readily derived from that obtained in Section V-C by simply considering the additional singular hypersurface corresponding to the platform attachment of the sixth leg that lies on the base plane.

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