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Analysing the Singularities of 6-SPS Parallel Robots Using Virtual Legs

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Abstract: *A virtual leg in a 6-SPS parallel robot is defined as a leg whose length is determined by the lengths of a subset of the actual legs of the robot. This necessarily implies that this subset of legs defines a rigid subassembly. In this paper, we consider four different rigid subassemblies, and show how the singularities of a robot containing one or several of these subassemblies are modified when substituting its actual legs by virtual legs.*

1 Introduction

In general, substituting one leg in a 6-SPS parallel robot by another arbitrary leg modifies the location of the platform singularities in a rather unexpected way. Nevertheless, in those cases in which the considered platform contains rigid subassemblies, legs can be substituted so that the singularity locus is modified in a controlled way.

In this paper we will consider the four rigid subassemblies appearing in Fig. 1. They can be seen as subassemblies involving (a) a point and a line, (b) a point and a plane, (c) two lines, and (d) a line and a plane, attached either to the base or the platform. In what follows, we will refer to them as PtL, PtP, LL, and LP subassemblies, respectively.

This work studies the effect of substituting one leg in the above subassemblies by another leg, so that the involved points, lines or planes remain invariant. We show how this operation either renders the platform singularity locus invariant, introduce non-obvious architectural singularities, or even introduce new singularities without altering the existing ones. The proposed leg substitutions include the possibility of splitting the multiple spherical joints and hence their practical interest.

The classification of 6-SPS parallel manipulators on the basis of the rigid subassemblies they contain was addressed in [Kong and Gosselin(2000)]. Each class consists of all the manipulators obtained by adding to a given rigid subassembly the remaining legs up to 6 in all possible topological configurations. Note that the manipulators in a class have neither the same forward kinematics nor the same singularity structure. The current work, on the contrary, tries to come up with transformations of manipulators that preserve their singularities, thus opening up the possibility of classifying manipulators in families sharing the same singularity structure.

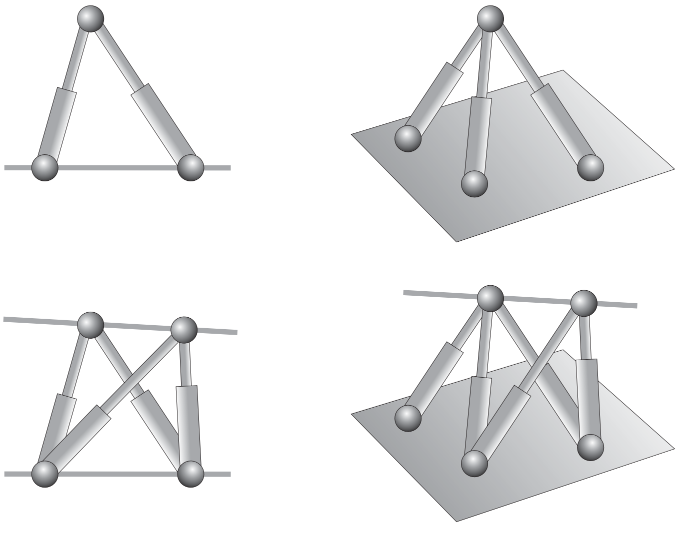

Figure 1: The four considered rigid subassemblies.

The paper is organized as follows. Section 2 presents how the Jacobian matrix of a 6-SPS parallel platform is modified by changing the location of one leg in those cases in which the length of this leg, in its new location, can be expressed in terms of the lengths of a subset of legs in their original locations. To make the presentation self-contained, Section 3 describes some basic facts concerning Cayley-Menger determinants and some of their properties needed in the subsequent sections. Sections 4, 5, 6, and 7 deal with the particular analysis of leg substitutions in each of the four considered rigid subassemblies. Section 8 presents an example showing how a 3-3 parallel robot has the same singularities as a 6-6 parallel robot by applying a sequence of the presented substitutions. Finally, Section 9 presents the conclusions.

2 Substituting actual legs by virtual legs

Let us consider a 6-SPS parallel platform whose six leg lengths are given by $l_1, ..., l_6$. Now, let us introduce a *virtual leg* whose length, say d, is implicitly determined by a function of the form:

$$f(l_1^2, l_2^2, \dots, l_6^2, d^2) = 0. \quad (1)$$

Then, it can be proved that

$$\frac{\partial d}{\partial l_i} = -\frac{l_i}{d} \cdot \frac{\partial f/\partial l_i^2}{\partial f/\partial d^2}. \quad (2)$$

As a consequence, the time derivative of d can be expressed as:

$$\dot{d} = -\sum_{i=1}^{6} \frac{l_i}{d} \cdot \frac{\partial f/\partial l_i^2}{\partial f/\partial d^2} \cdot \dot{l}_i. \quad (3)$$

Assuming that the Jacobian matrix that relates the leg length velocities, $\dot{l}_1, \dot{l}_2, \dots, \dot{l}_6$, with the end-effector velocity vector $(\mathbf{v}, \mathbf{\Omega})$ can be expressed as:

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{\Omega} \end{pmatrix} = J \begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \vdots \\ \dot{l}_6 \end{pmatrix}, \quad (4)$$

then singularities arise either when $\det(J) = 0$ or $\det(J) = \infty$.

Now, let us assume, without loss of generality, that leg 1 is substituted by the virtual leg. Then,

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{\Omega} \end{pmatrix} = J\, L^{-1} \begin{pmatrix} \dot{d} \\ \dot{l}_2 \\ \vdots \\ \dot{l}_6 \end{pmatrix},$$

where

$$L = \begin{pmatrix} \frac{l_1}{d}\frac{\partial f/\partial l_1^2}{\partial f/\partial d^2} & \frac{l_2}{d}\frac{\partial f/\partial l_2^2}{\partial f/\partial d^2} & \dots & \frac{l_6}{d}\frac{\partial f/\partial l_6^2}{\partial f/\partial d^2} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \quad (5)$$

We conclude that the singularities of the parallel robot in which leg 1 has been substituted by the virtual leg are those configurations in which the term:

$$\det(J)\det(L^{-1}) = \frac{\det(J)}{\det(L)} = \det(J)\frac{d}{l_1}\frac{\partial f/\partial d^2}{\partial f/\partial l_1^2} \quad (6)$$

is either 0 or ∞. This result has important consequences. For example, if in the working space of the robot $\frac{d}{l_1}\frac{\partial f/\partial d^2}{\partial f/\partial l_1^2}$ is always different from 0 and ∞, the introduced substitution leaves the singularities of the original robot invariant. On the contrary, if $\frac{d}{l_1}\frac{\partial f/\partial d^2}{\partial f/\partial l_1^2}$ is identically zero, the substitution is introducing an architectural singularity.

In practice, we will be interested in repeating this kind of substitution a number of times. Then, a sequence of terms of the form $\frac{d_i}{l_j}\frac{\partial f/\partial d_i^2}{\partial f/\partial l_j^2}$ will appear multiplying the determinant of the original robot Jacobian. In what follows, each of these terms will be called *singularity factor*. Since poles and zeros in the resulting sequence of singularity factors may cancel, singularities may be added and removed at each step.

3 Cayley-Menger determinants

Let us define

$$D(i_1, \dots, i_n; j_1, \dots, j_n) = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & s_{i_1,j_1} & \dots & s_{i_1,j_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{i_n,j_1} & \dots & s_{i_n,j_n} \end{vmatrix}, \quad (7)$$

with $s_{i,j}^2 = \|\mathbf{p}_i - \mathbf{p}_j\|^2$. This determinant is known as the *Cayley-Menger bi-determinant* of the point sequences $\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_n}$, and $\mathbf{p}_{j_1}, \dots, \mathbf{p}_{j_n}$. When the two point sequences are the same, it will be convenient to abbreviate $D(i_1, \dots, i_n; i_1, \dots, i_n)$ by $D(i_1, \dots, i_n)$, which is simply called the *Cayley-Menger determinant* of the involved points.

The square volume $V^2(\mathbf{p}_0, \dots, \mathbf{p}_k)$ of the $k-$dimensional simplex defined by the $k+1$ points $\mathbf{p}_0, \dots, \mathbf{p}_k$ can be expressed as follows:

$$V^2(\mathbf{p}_0, \dots, \mathbf{p}_k) = \frac{(-1)^{k+1}}{2^k (k!)^2} D(0, \dots, k). \quad (8)$$

Two properties of these determinants that will be useful later are [Thomas and Ros (2005)]:

$$D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = ((\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3)) \cdot ((\mathbf{q}_1 - \mathbf{q}_3) \times (\mathbf{q}_2 - \mathbf{q}_3)), \quad (9)$$

and

$$D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4; \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = |\mathbf{p}_1 - \mathbf{p}_4, \mathbf{p}_2 - \mathbf{p}_4, \mathbf{p}_3 - \mathbf{p}_4| \cdot |\mathbf{q}_1 - \mathbf{q}_4, \mathbf{q}_2 - \mathbf{q}_4, \mathbf{q}_3 - \mathbf{q}_4|. \quad (10)$$

4 Leg substitutions in PtL subassemblies

Let us consider the PtL subassembly and the virtual leg shown in Fig. 2.

Since the tetrahedron defined by points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ has null volume, then

$$D(1,2,3,4) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (m+n)^2 & l_1^2 & m^2 \\ 1 & (m+n)^2 & 0 & l_2^2 & n^2 \\ 1 & l_1^2 & l_2^2 & 0 & d^2 \\ 1 & m^2 & n^2 & d^2 & 0 \end{vmatrix} = 0.$$

In other words,

$$n l_1^2 + m l_2^2 - (m+n)d^2 - mn(m+n) = 0. \quad (11)$$

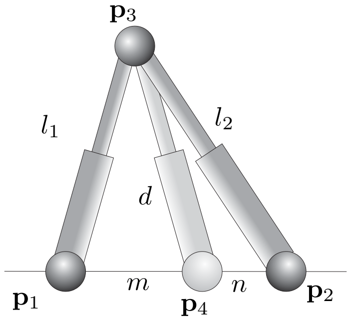

Figure 2: A PtL subassembly. The leg in light grey represents a virtual leg.

Thus, using (3), the time derivative of the virtual leg length can be expressed as:

$$\dot{d} = \frac{l_1}{d}\frac{n}{m+n}\dot{l}_1 + \frac{l_2}{d}\frac{m}{m+n}\dot{l}_2. \tag{12}$$

Then,

$$\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \vdots \\ \dot{l}_6 \end{pmatrix} = \begin{pmatrix} \frac{l_1}{d}\frac{n}{m+n} & \frac{l_2}{d}\frac{m}{m+n} & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} \dot{d} \\ \dot{l}_2 \\ \vdots \\ \dot{l}_6 \end{pmatrix}. \tag{13}$$

The determinant of the above inverse matrix is equal to

$$\frac{d}{l_1}\frac{m+n}{n}. \tag{14}$$

Due to the fact that $m+n \neq 0$, since otherwise the two legs in the original robot would be coincident, a singularity is introduced only if $n = 0$.

Note that the above derivation can be greatly simplified by directly analysing the zeros of the partial derivatives in (6). We will proceed in this way in the following sections.

5 Leg substitutions in PtP subassemblies

Let us consider the PtP subassembly and the virtual leg appearing in Fig. 3.

The five points $\mathbf{p_1}$,..., $\mathbf{p_5}$ define a simplex in $\mathbb{R}^4$ but, since it is embedded in $\mathbb{R}^3$, its volume is null. The equation $D(1,2,3,4,5) = 0$ can be simplified by applying Jacobi's theorem to the following partition of D(1, 2, 3, 4, 5)

$$\left|\begin{array}{cccc|cc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & m_1^2 & m_3^2 & l_1^2 & p_1^2 \\ 1 & m_1^2 & 0 & m_2^2 & l_2^2 & p_2^2 \\ 1 & m_2^2 & m_2^2 & 0 & l_3^2 & p_3^2 \\ \hline 1 & l_1^2 & l_2^2 & l_3^2 & 0 & d^2 \\ 1 & p_1^2 & p_2^2 & p_3^2 & d^2 & 0 \end{array}\right|,$$

where $p_i = d(p_i, p_5)$. Then, $D(1,2,3,4,5) = 0$ yields

$$\frac{D(1,2,3,4)D(1,2,3,5) - D^2(1,2,3,4;1,2,3,5)}{D(1,2,3)} = 0,$$

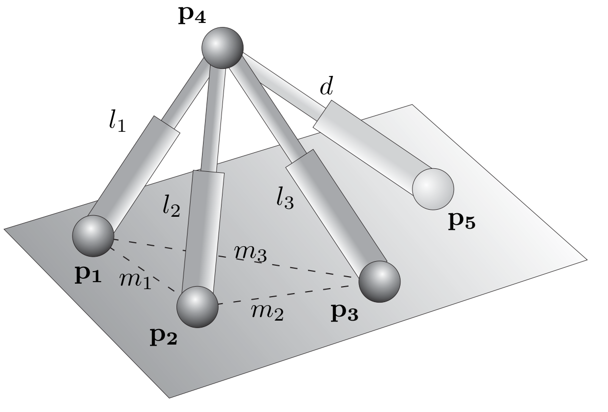

Figure 3: An PtP subassembly. The leg in light gray represents a virtual leg.

but $D(1,2,3,5) = 0$ because the four points $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{p_3}$ and $\mathbf{p_5}$ are in the same plane. Thus, we get the following linear equation in d^2:

$$D(1,2,3,4;1,2,3,5) = 0. \tag{15}$$

Now, deriving $D(1,2,3,4;1,2,3,5)$ with respect to d^2 and l_1^2, using (6), we get:

$$\begin{aligned} \frac{\partial f}{\partial d^2} =& D(1,2,3), \quad \text{and} \\ \frac{\partial f}{\partial l_1^2} =& -D(1,2,3;2,3,5). \end{aligned} \tag{16}$$

Note that, using (8) and (9), such determinants can be expressed as:

$$\begin{aligned} &D(1,2,3) = 4A(1,2,3)^2 \quad \text{and} \\ &D(1,2,3;2,3,5) = 2A(1,2,3)2A(2,3,4), \end{aligned} \tag{17}$$

where $A(i,j,k)$ stands for the area of the triangle defined by $\mathbf{p}_i$, $\mathbf{p}_j$ and $\mathbf{p}_k$.

Then, the singularity factor is:

$$\frac{d}{l_1}\frac{A(1,2,3)}{A(2,3,5)}, \tag{18}$$

that is, the area of the old triangular base divided by the area of the new triangular base, which is a constant value. Thus, factor (18) does not introduce any singularity provided that $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_5$ are not collinear (assuming that the initial $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ were not collinear either).

It can be proved that this leg substitution can be reduced to two consecutive leg substitutions in PtL subassemblies.

6 Leg substitutions in LL subassemblies

Let us consider the LL subassembly and the virtual leg appearing in Fig. 4.

Points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_6$ define a simplex in $\mathbb{R}^5$ but, since it is embedded in $\mathbb{R}^3$, its volume is null. Hence, $D(1,2,3,4,5,6) = 0$. This defines a quadratic equation in $s_{5,6}$, the length of the

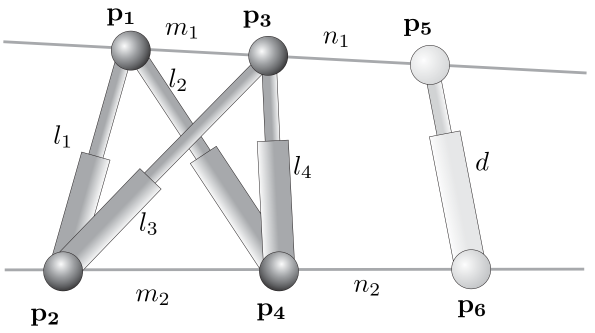

Figure 4: An LL subassembly. The leg in light gray represents a virtual leg.

virtual leg. This equation can be simplified by applying Jacobi's theorem to the following partition of D(1, 2, 3, 4, 5, 6)

$$\left|\begin{array}{ccccc|cc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_1^2 & m_1^2 & l_2^2 & p_1^2 & s_{1,6} \\ 1 & l_1^2 & 0 & l_3^2 & m_2^2 & s_{2,5} & p_2^2 \\ 1 & m_1^2 & l_3^2 & 0 & l_4^2 & n_1^2 & s_{3,6} \\ 1 & l_2^2 & m_2^2 & l_2^2 & 0 & s_{4,5} & n_2^2 \\ \hline 1 & p_1^2 & s_{2,5} & n_1^2 & s_{5,4} & 0 & d^2 \\ 1 & s_{6,1} & p_2^2 & s_{6,3} & n_2^2 & d^2 & 0 \end{array}\right|,$$

where $p_1 = m_1 + n_1$ and $p_2 = m_2 + n_2$. Then, we conclude that $D(1, 2, 3, 4, 5, 6) = 0$ yields

$$\frac{D(1,2,3,4,5)D(1,2,3,5,6) - D^2(1,2,3,4,5;1,2,3,5,6)}{D(1,2,3,4)} = 0.$$

Now, note that $D(1, 2, 3, 4, 5) = 0$ and $D(1, 2, 3, 5, 6) = 0$ because they correspond to the volumes of simplices in $\mathbb{R}^4$. Thus, assuming that the tetrahedron defined by $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ is not degenerate,

$$D(1, 2, 3, 4, 5; 1, 2, 3, 5, 6) = 0, \quad (19)$$

which is linear in d^2. By deriving this implicit equation with respect to d^2 and l_1^2, we obtain

$$\begin{aligned} \frac{\partial f}{\partial d^2} =& D(1,2,3,4) \\ \frac{\partial f}{\partial l_1^2} =& -\left(D(1,3,4,5;2,3,4,6) + D(2,3,4,5;1,3,4,6)\right) \\ & - \frac{n_2}{m_2} D(2,3,4,5;1,2,3,4) \\ & + \frac{n_1}{m_1} D(1,2,3,4;1,3,4,6). \end{aligned} \quad (20)$$

where the unknown squared distances, $s_{i,j}$, can be readily obtained using substitutions in PtL subassemblies [equation (11)].

Since $D(1, 3, 4, 5; 2, 3, 4, 6) = 0$ because the volume defined by $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{p}_4$ and $\mathbf{p}_5$ is null [equation (10)], it can be proved that the second partial derivative in (20) is:

$$\frac{\partial f}{\partial l_1^2} = -\frac{n_1 n_2}{m_1 m_2} D(1, 2, 3, 4) \quad (21)$$

so the singularity factor for the LL substitution is

$$-\frac{d}{l_1} \frac{m_1 m_2}{n_1 n_2} \quad (22)$$

Since the singularity factor is neither zero nor infinite, no new singularity is introduced.

Notice that, by introducing leg substitutions in an LL subassembly and leg substitutions in the PtL subassemblies contained in it, we can obtain any configuration of legs connecting arbitrary points in both lines.

7 Leg substitutions in LP subassemblies

Let us consider the LP subassembly and the virtual leg shown in Fig. 5.

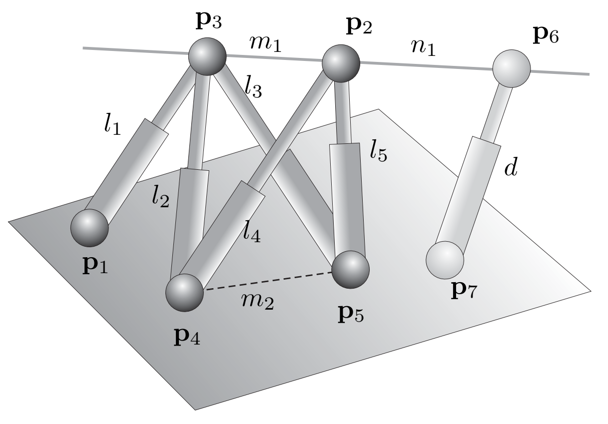

Figure 5: A LP subassembly. The leg in light grey represents a virtual leg.

In this case, let us consider points $\mathbf{p}_3, \ldots, \mathbf{p}_7$. These points can be seen as two pyramids with known edge lengths sharing the same triangular base so that the distance between their apexes, $\mathbf{p}_6$ and $\mathbf{p}_7$, is the length of the virtual leg. Clearly, there are two solutions for this length. These five points define a simplex in $\mathbb{R}^4$ but, since it is embedded in $\mathbb{R}^3$, its volume is null. Hence, $D(3, 4, 5, 6, 7) = 0$. This defines a quadratic equation in $s_{6,7}$ that can be simplified by applying Jacobi's theorem to the following partition of $D(3, 4, 5, 6, 7)$

$$\left|\begin{array}{cccc|cc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_2^2 & l_3^2 & q^2 & s_{3,7} \\ 1 & l_2^2 & 0 & m_2^2 & s_{4,6} & p_4^2 \\ 1 & l_3^2 & m_2^2 & 0 & s_{5,6} & p_5^2 \\ \hline 1 & q^2 & s_{6,4} & s_{6,5} & 0 & d^2 \\ 1 & s_{7,3} & p_4^2 & p_5^2 & d^2 & 0 \end{array}\right|,$$

where $p_i = d(p_i, p_7)$ and $q = (m_1 + n_1)$, concluding that $D(3, 4, 5, 6, 7) = 0$ yields

$$\frac{D(3,4,5,6)D(3,4,5,7) - D^2(3,4,5,6;3,4,5,6)}{D(3,4,5)} = 0.$$

Assuming that the triangle defined by $\mathbf{p}_3$, $\mathbf{p}_4$, and $\mathbf{p}_5$ is not degenerate, then

$$D(3, 4, 5, 6; 3, 4, 5, 7)^2 - D(3, 4, 5, 6)D(3, 4, 5, 7) = 0. \quad (23)$$

Since this equation is quadratic in d^2, there exist two possible solutions for the length of the virtual leg, as expected. This means that this substitution will necessarily introduce new singularities.

The partial derivative of (23) with respect to d^2 can be expressed as:

$$\frac{\partial f}{\partial d^2} = 2D(3,4,5)D(3,4,5,6;3,4,5,7) \tag{24}$$

The partial derivative of (23) with respect to l_1 is a bit more complex. In (23), l_1 only appears in the computation of $s_{3,7}$ which can be computed using a PtP substitution. To this end, let us consider the PtP subassembly formed by points $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{p}_4$ and $\mathbf{p}_5$. Distance $s_{3,7}$ can be computed by expanding equation (15) by its minors yielding

$$s_{3,7} = \frac{l_1^2 D(145;457) - l_2^2 D(145;157) + l_3^2 D(145;147)}{D(145)}. \tag{25}$$

where commas between indices have been removed to ease notation.

Now, applying the chain rule, we get

$$\frac{\partial f}{\partial l_1^2} = \frac{\partial f}{\partial s_{3,7}} \frac{\partial s_{3,7}}{\partial l_1^2} \tag{26}$$

where

$$\frac{\partial s_{3,7}}{\partial l_1^2} = \frac{D(145;457)}{D(145)}, \tag{27}$$

and

$$\begin{aligned}\frac{\partial f}{\partial s_{3,7}} =& -2D(3456;3457)D(456;345) \\ &+ D(3456)D(345;457).\end{aligned} \tag{28}$$

Then, multiplying and simplifying the result using (9) and (10), we get

$$\frac{d}{l_1} \frac{2D^{1/2}(345)}{2|3457|D^{1/2}(456)\cos(\phi_1) + |3456|D^{1/2}(457)\cos(\phi_2)}, \tag{29}$$

where ϕ_1 and ϕ_2 are the dihedral angles between plane $\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5$ and planes $\mathbf{p}_4\mathbf{p}_5\mathbf{p}_6$ and $\mathbf{p}_4\mathbf{p}_5\mathbf{p}_7$, respectively, and $|ijks| = |\mathbf{p}_i - \mathbf{p}_s, \mathbf{p}_j - \mathbf{p}_s, \mathbf{p}_k - \mathbf{p}_s|$.

Note that this result would have been quite different if we had substituted the second leg instead of the first one by the virtual leg, as an LP subassembly is not symmetric. If we want to substitute the second leg, we must compute the partial derivative of (23) with respect to l_2, which is much more complicated because l_2 appears in the computation of $s_{3,7}$, $s_{4,6}$, and directly in the implicit function. The result is not included here, but the derivation unfolds analogously to the one presented above for the substitution of the first leg.

8 Example

Let us consider the 6-SPS parallel robot in Fig. 6(top). By applying several leg substitutions in the PtL and LL subassemblies it contains, it will be shown that the Jacobian determinant of this robot and that in Fig. 6(bottom) is the same except for a constant factor that only depends on fixed metric distances between the attachment points.

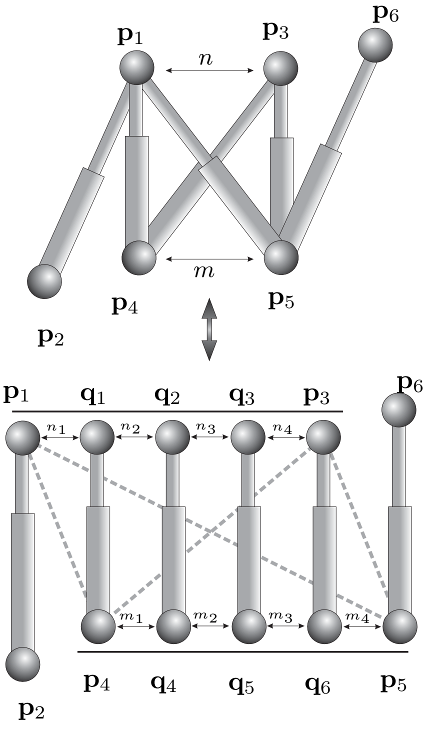

Figure 6: A 3-3 parallel robot containing eight PtL subassemblies and one LL subassembly (top), and the resulting robot after performing four leg substitutions in their PtL subassemblies and two leg substitutions in its LL subassembly (bottom).

Using Grassmann-Cayley algebra, it can be shown that the determinant of the Jacobian of the 6-SPS parallel robot in Fig. 6(top) can be expressed as [Downing, Samuel and Hunt(2002)]:

$$l_1 l_2 l_3 l_4 l_5 l_6 [1245][1345][1356]. \tag{30}$$

where $[ijkl]$ is a *bracket*, a mathematical entity that in this case can be interpreted as the volume of the tetrahedron defined by points i, j, k and l [Ben-Horin and Shoham(2007)].

Now, let us introduce the following points:

$$\begin{aligned}\mathbf{q}_i =& k_i \mathbf{p}_1 + (1-k_i)\mathbf{p}_3, \text{ for } i=1,2,3 \\ \mathbf{q}_j =& k_j \mathbf{p}_4 + (1-k_j)\mathbf{p}_5, \text{ for } j=4,5,6\end{aligned} \tag{31}$$

where

$$k_i = \frac{\sum_{r=1}^{i} n_r}{n}, \qquad k_{i+3} = \frac{\sum_{r=1}^{i} m_r}{m}, \tag{32}$$

n_i and m_i satisfying $\sum_{i=1}^{4} n_i = n$ and $\sum_{i=1}^{4} m_i = m$, as depicted in Fig. 6(bottom).

After performing four leg substitutions in PtL subassemblies and two leg substitutions in the LL subassembly, in the configuration shown in 6(top), it is possible to obtain the configuration of legs shown in Fig. 6(bottom). After multiplying and simplifying all resulting singularity factors, the Jacobian determinant of the resulting 6-6 platform can be expressed as:

$$
\begin{aligned}
& l_1 d_2 d_3 d_4 d_5 l_6 \cdot \\
& \left(\frac{(n_2+n_3)(n_3+n_4)(m_1+m_2+m_3)m_2}{n^2+m^2} - \right. \\
& \left. \frac{(n_2+n_3+n_4)n_3(m_1+m_2)(m_2+m_3)}{n^2+m^2} \right) . \\
& [1245][1345][1356]
\end{aligned}
\tag{33}
$$

where $d_i, i = 2, \cdots, 5$ are the lengths of the new legs. Notice that legs 1 and 6 have not been changed. It can be checked that the resulting product of singularity factors is zero if, and only if, the cross-ratios of the upper and lower aligned points is equal (see [Borràs *et al.*(2008)] for details). Thus, the performed substitutions lead to an architectural singularity when this cross-ratio relation is satisfied.

This architectural singularity is the one leading to the *LL* singular subassembly studied in [Kong (1998)], and it also appears as the fifth type of singularity in Theorem 1 of [Husty and Karger (2000)].

9 Conclusions

We have shown how the singularities of a robot containing rigid subassemblies are modified when substituting its actual legs by virtual legs that leave invariant one of these subassemblies at each substitution.

Using the presented approach based on Cayley-Menger determinants, it seems also feasible to accommodate the analysis of leg substitutions in plane-plane subassemblies, that is, a whole 6-SPS parallel robot with planar base and platform. This certainly deserves further attention. The resulting symbolic expression for this general substitution will probably be quite complex, but its attainment would provide the maximum generality to the presented approach.

The idea of singularity-preserving transformations put forth in this paper opens up the possibility of classifying parallel platforms in families sharing the same singularity structure, as was done for flagged manipulators in [Alberich-Carramiñana *et al.*(2007)].

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